Math 7200, Problem Set I

Nitesh Mathur

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1. Show that $(B(H), ||\cdot||_\infty)$ is a complete space.

Definitions 1. A sequence $\{x_n\}$ in a metric space (X, d) is **Cauchy** if $\forall\ \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for $m, n > N, d(x_m, x_n) = ||x_m - x_n|| < \epsilon$.

1b. Note that every Cauchy sequence in a metric space is bounded.

2. A metric space (X, d) is **complete** if every Cauchy sequence in X converges in X.

Solution *Proof.* First recall that $B(H) = \{T : H \to H|$ linear, bounded $\}$, where H is a Hilbert space. Note that $||T||_\infty = \sup_{||\xi|| \leq 1} ||T(\xi)||$.

Pick a Cauchy sequence T_n in T. Since it is Cauchy, by definition we have $\forall \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for $m, n > N$, $||T_n - T_m|| = \sup_{||\xi|| \leq 1||} ||T_n(\xi) - T_m(\xi)|| < \epsilon$.

We need to show there is some $T \in B(H)$ such that $\lim_{n\to\infty} ||T_n - T|| = 0$.

$$\begin{aligned}
||T_n - T_m|| &= \sup_{||\xi|| \leq 1||} ||T_n(\xi) - T_m(\xi)|| \\
&< \epsilon \\
\Rightarrow ||T_n - T|| &= ||T_n - T_m + T_m - T|| \\
&\leq ||T_n - T_m|| + ||T - T_m|| \\
&= \underbrace{\sup_{||\xi|| \leq 1||} ||T_n(\xi) - T_m(\xi)||}_{\text{(Converges since Cauchy)}} + \sup_{||\xi|| \leq 1||} ||T(\xi) - T_m(\xi)||
\end{aligned}$$

Since T is linear and bounded, by taking limits on both sides, we see that $T_n \to T \in B(H)$ as $m \to \infty$. ■

2. Fix $I \in M(A) \to A/I$ (This is a field since I is maximal).

Suppose $(A, ||\cdot||)$ is a Banach Algebra. $\bar{I} = I \leq A$ is a closed ideal $\Rightarrow (A/I, ||\cdot$
$_I)$ is a Banach algebra, where

$$||\alpha + I||_I = \inf_{x \in \lambda} ||\alpha + x||_A \leq ||a||.$$

Solution We need to show that A is a normed space, is sub-multiplicative, and is complete.

(i) Show $||c \cdot x|| = |c| \cdot ||x|| \; \forall x \in A, c \in \mathbb{C}$.

Proof. Let $\alpha + I \in A/I$ and $c \in \mathbb{C}$. Then, we have

$$\begin{aligned} ||c \cdot (\alpha + I)|| &= |c\alpha + I|| \\ &= \inf_{x \in I} ||ca + cx||_A \\ &= \inf_{x \in I} ||c(a + x)||_A \\ &= |c| \cdot \inf_{x \in I} ||\alpha + x||_A \text{ (since } A \text{ is a Banach Algebra)} \\ &= |c| \cdot ||\alpha + I|| \end{aligned}$$

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(ii) Show $||x + y|| \leq ||x|| + ||y||$.

Proof. Let $\alpha + I, \beta + I \in A/I$. Then,

$$\begin{aligned} ||\alpha + I + \beta + I|| &= ||(\alpha + \beta) + I|| \\ &= \inf_{x \in I} ||(\alpha + \beta) + x||_A \\ &= \inf_{x \in I} ||\alpha + x||_A + \inf_{x \in I} ||\beta + x||_A \text{ (since } A \text{ is a Banach Algebra)} \\ &\leq ||\alpha + I|| + ||\beta + I|| \end{aligned}$$

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(iii) Show $||x|| = 0 \iff x = 0_A$.

Proof. Let $\alpha + I \in A/I$. Then, we have

$$\begin{aligned} &||\alpha + I|| = 0 \\ \Rightarrow &\inf_{x \in I} ||\alpha + x||_A \leq ||\alpha|| = 0 \\ &\iff \alpha = 0_A \text{ (since } A \text{ is a Banach Algebra)} \\ &\iff \alpha + I = 0_{A/I} \end{aligned}$$

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(iv) Show sub-multiplicative i.e. $||xy|| \leq ||x|| \cdot ||y||$.

Proof. Let $\alpha + I, \beta + I \in A/I$. Then,

$$\begin{aligned} ||(\alpha + I)(\beta + I)|| &= ||\alpha\beta + I|| \\ &= \inf_{x \in I} ||\alpha\beta + x||_A \\ &\leq ||\alpha\beta|| \\ &\leq ||\alpha|| \cdot ||\beta|| \text{ (since } A \text{ is a Banach Algebra)} \\ &= \inf_{x \in I} ||\alpha + x||_A \cdot \inf_{x \in I} ||\beta + x||_A \\ &\leq ||\alpha + I|| \cdot ||\beta + I|| \end{aligned}$$

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(v) Show that $(A, ||\cdot||)$ is complete, i.e. all $||\cdot||$-Cauchy sequences converge.

Proof. Suppose $(\alpha + I)_n$ is a Cauchy sequence in A/I i.e. $\forall\ \epsilon > 0,\ \exists N \in \mathbb{N}$ such that for all $m, n > N$, $||(\alpha + I)_n - (\alpha + I)_m|| < \epsilon$. We need to show that this sequence converges *in* A/I. Note by definition,

$||(\alpha + I)_n - (\alpha + I)_m|| = \inf_{x \in I} ||(\alpha + x)_n - (\alpha + x)_m||_A$.

Since A is a Banach algebra, every Cauchy sequence of A converges in A. On the other, we assume that $||(\alpha + I)_n - (\alpha + I)_m|| < \epsilon$ and by property (ii), the norm of the sums is bounded by their sum of the norms. Combining these two facts and taking the infimum on the right hand side, we have that $(A/I, ||\cdot||)$ is a complete space. ■

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3. Show that $\{f \in C(K) | f(K) = 0 \text{ continuous}\}$ is a maximal ideal of $C(K) = \{f : K \to \mathbb{C} \text{ continuous}\}$, where K is a Hausdorff compact space.

Know 1. There exists a natural homomorphism:

$$K \to \sigma(C(K))$$
$$k \mapsto \phi_k$$

defined by $\phi_k(f) = f(k)\ \forall f \in C(K)$. This map is a homeomorphism.

2. A two-sided ideal I of a ring R is both a left ideal and a right ideal i.e. it is a subring, $rI \subset I$, and $Ir \subset I$ for all $r \in I$. (absorbs products).

Solution *Proof.* We first recall that M is a maximal ideal iff the quotient ring $C(K)/M$ is a field.

By the First Isomorphism Theorem for rings, $C(K)/\ker(\phi_k) \cong \mathbb{C}$.

In particular, the kernel of ϕ is the set of all continuous functions such that $\phi_k(f) = f(k) = 0\} = M$.

Since $\mathbb{C}$ is a field, $C(K)/\ker(\phi_k) = C(K)/M$ is a field $\Rightarrow$ M is a maximal ideal. ■

Alternatively (Assume that this is not a maximal ideal, i.e. $\exists\ J$ ideal such that $I \subset J \subset C(K)$, where $I = \{f \in C(K) | f(K) = 0 \text{ continuous }\}$.

Suppose $f_1 \in J$ such that $f_1 \notin I$. Hence, we have $1 = \dfrac{f_1(k) - f_1 + f_1}{f_1(k)}$ for some k. Since I is an ideal the first part of the sum goes to 0. Hence, we have $1 = \dfrac{f_1}{f_1(k)} \Rightarrow 1 \in J$ and so $J = C(K)$, the entire space. Therefore, $I = M$ is a maximal ideal).